On Compatibility of Discrete Relations

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Abstract. An approach to compatibility analysis of systems of discrete relations is proposed. Unlike the Gröbner basis technique, the proposed scheme is not based on the polynomial ring structure. It uses more primitive set-theoretic and topological concepts and constructions. We illustrate the approach by application to some two-state cellular automata. In the two-state case the Gröbner basis method is also applicable, and we compare both approaches.

1 Introduction

A typical example of a system of discrete relations is a cellular automaton. Cellular automata are used successfully in a large number of applications.[1] Furthermore, the concept of cellular automaton can be generalized, and we consider the following extension of the standard notion of a cellular automaton:

1. Instead of regular uniform lattice representing the space and time in a cellular automaton, we consider more general *abstract simplicial complex* $K = (X, \Delta)$ (see, e.g., [2]). Here $X = \{x_0, x_1, \ldots\}$ is a finite (or countably infinite) set of *points*; Δ is a collection of subsets of X such that (a) for all $x_i \in X,\ \{x_i\} \in \Delta$; (b) if $\tau \subseteq \delta \in \Delta$, then $\tau \in \Delta$.
   The sets $\{x_i\}$ are called *vertices*. We say $\delta \in \Delta$ is a $k-$*simplex* of *dimension* k if $|\delta| = k+1$, i.e., $\dim\delta = |\delta| - 1$. The *dimension of complex* K is defined as the maximum dimension of its constituent simplices $\dim K = \max\limits_{\delta\in\Delta} \dim\delta$.
   If $\tau \subseteq \delta$, τ is called a *face* of δ. Since any face of a simplex is also a simplex, the topological structure of the complex K, i.e., the set Δ is uniquely determined by the set of *maximal simplices* under inclusion.
   One of the advantages of simplicial complexes over regular lattices is their applicability to models with dynamically emerging and evolving rather than pre-existing space-time structure.

[1] Comparing expressiveness of cellular automata and differential equations, T. Toffoli writes [1]: "Today, it is clear that we can do all that differential equations can do, and more, because it is differential equations that are the poor man's cellular automata — not the other way around!"

2. The dynamics of a cellular automaton is determined by a *local rule*

$$x_{i_k} = f\left(x_{i_0}, \ldots, x_{i_{k-1}}\right). \tag{1}$$

In this formula $x_{i_0}, \ldots, x_{i_k} \in X$ are interpreted as discrete variables taking values in a finite set of states S canonically represented as

$$S = \{0, \ldots, q-1\}.$$

The set of points $\left\{x_{i_0}, \ldots, x_{i_{k-1}}\right\}$ is called the *neighborhood*. The point x_{i_k} is considered as the "next time step" match of some point, say $x_{i_{k-1}}$, from the neighborhood.
A natural generalization is to replace function (1) by a *relation* on the set $\{x_{i_0}, \ldots, x_{i_k}\}$. In this context, local rule (1) is a special case of relation. Relations like (1) are called *functional relations*. They are too restrictive in many applications. In particular, they violate in most cases the symmetry among points $x_{i_0}, \ldots, x_{i_k}$. Furthermore, we will see below that the functional relations, as a rule, have non-functional consequences.

We can formulate some natural problems concerning the above structures:

1. *Construction of consequences.* Given a relation R^δ on a set of points δ, construct non-trivial relations R^τ on subsets $\tau \subseteq \delta$, such that $R^\delta \Rightarrow R^\tau$.
2. *Extension of relation.* Given a relation R^τ on a subset $\tau \subseteq \delta$, extend it to relation R^δ on the superset δ.
3. *Decomposition of relation.* Given a relation R^δ on a set δ, decompose R^δ into combination of relations on subsets of δ.
4. *Compatibility problem.* Given a collection of relations $\left\{R^{\delta_1}, \ldots, R^{\delta_n}\right\}$ defined on sets $\{\delta_1, \ldots, \delta_n\}$, construct relation $R^{\cup_{i=1}^n \delta_i}$ on the union $\bigcup_{i=1}^n \delta_i$, such that $R^{\cup_{i=1}^n \delta_i}$ is compatible with the initial relations.
5. *Imposing topological structure.* Given a relation R^X on a set X, endow X with a structure of simplicial complex consistent with the decomposition of the relation.

If the number of states is a power of a prime, i.e., $q = p^n$, we can always[2] represent any relation over k points $\{x_1, \ldots, x_k\}$ by the set of zeros of some polynomial from the ring $\mathbb{F}_q[x_1, \ldots, x_k]$ and study the compatibility problem by the standard Gröbner basis methods. It would be instructive to look at the compatibility problem from the set-theoretic point of view cleared of the ring structure influence.

An example from fundamental physics is the *holographic principle* proposed by G. 't Hooft and developed by many authors (see [4,5]). According to 't Hooft the combination of quantum mechanics and gravity implies that the world at the Planck scale can be described by a three-dimensional discrete lattice theory with a spacing of the Planck length order. Moreover, a full description of events on

[2] Due to the functional completeness of polynomials over $\mathbb{F}_q$ (see [3]) any function mapping k elements of $\mathbb{F}_q$ into $\mathbb{F}_q$ can be realized by a polynomial.

the three-dimensional lattice can be derived from a set of Boolean data (one bit per Planck area) on a two-dimensional lattice at the spatial (evolving with time) boundaries of the world. The transfer of data from two to three dimensions is performed in accordance with some local relations (constraints or laws) defined on plaquettes of the lattice. Since the data on points of the three-dimensional lattice are overdetermined, the control of compatibility of relations is necessary. Large number of constraints compared to the freedom one has in constructing models is one of the reasons why no completely consistent mathematical models describing physics at the Planck scale have been found so far.

2 Basic Definitions and Constructions

The definition of *abstract* k-simplex as a set of $k+1$ points is motivated by the fact that $k+1$ points generically embedded in Euclidean space of sufficiently high dimension determine k-dimensional convex polyhedron. The abstract combinatorial topology only cares about how the simplices are connected, and not how they can be placed within whatever spaces.[3] We need to consider also k-point sets which we call k-*sets*. Notice that k-sets may or may not be $(k-1)$-simplices.

A relation is defined as a subset of a Cartesian product $S \times \cdots \times S$ of the set of states. Dealing with the system of relations determined over different sets of points we should indicate the correspondence between points and dimensions of the hypercube $S \times \cdots \times S$. The notation $S^{\{x_i\}}$ specifies the set S as a set of values for the point x_i. For the k-set $\delta = \{x_1, \ldots, x_k\}$ we denote $S^\delta \equiv S^{\{x_1\}} \times \cdots \times S^{\{x_k\}}$.

A ***relation*** R^δ over a k-set $\delta = \{x_1, \ldots, x_k\}$ is any subset of the hypercube S^δ, i.e., $R^\delta \subseteq S^\delta$. We call the set δ *domain* of the relation R^δ. The relations $\emptyset^\delta$ and S^δ are called *empty* and *trivial*, respectively.

Given a set of points δ, its subset $\tau \subseteq \delta$ and relation R^τ over the subset τ, we define ***extension*** of R^τ as the relation

$$R^\delta = R^\tau \times S^{\delta \setminus \tau}.$$

The procedure of extension allows one to extend relations $R^{\delta_1}, \ldots, R^{\delta_m}$ defined on different domains to the common domain, i.e., the union $\delta_1 \cup \cdots \cup \delta_m$.

Now we can construct the ***compatibility condition*** of the system of relations $R^{\delta_1}, \ldots, R^{\delta_m}$. Naturally this is intersection of extensions of the relations to the common domain

$$R^\delta = \bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta \setminus \delta_i} \right), \text{ where } \delta = \bigcup_{i=1}^{m} \delta_i.$$

We call the compatibility condition R^δ the ***base relation*** of the system of relations $R^{\delta_1}, \ldots, R^{\delta_m}$. If the base relation is empty, the relations $R^{\delta_1}, \ldots, R^{\delta_m}$ are *incompatible*. Note that in the case $q = p^n$ the compatibility condition can

[3] There are mathematical structures of non-geometric origin, like *hypergraphs* or *block designs*, closely related conceptually to the abstract simplicial complexes.

be represented by a single polynomial, in contrast to the Gröbner basis approach (of course, the main aim of the Gröbner basis computation — construction of basis of polynomial ideal — is out of the question).

A relation Q^δ is a *consequence* of relation R^δ, if $R^\delta \subseteq Q^\delta \subseteq S^\delta$, i.e., Q^δ is any superset of R^δ. Any relation can be represented in many ways by intersections of different sets of its consequences:

$$R^\delta = Q^{\tau_1} \cap \cdots \cap Q^{\tau_r}.$$

We call such representations *decompositions*.

In the polynomial case $q = p^n$, any possible Gröbner basis of polynomials representing the relations $R^{\delta_1}, \ldots, R^{\delta_m}$ corresponds to some decomposition of the base relation R^δ of the system $R^{\delta_1}, \ldots, R^{\delta_m}$. However, the decomposition implied by a Gröbner basis may look accidental from our point of view and if $q \neq p^n$ such decomposition is impossible at all.

The total number of all consequences (including R^δ itself and the trivial relation S^δ) is, obviously,

$$2^{\left(q^k - \left|R^\delta\right|\right)}.$$

In our context it is natural to distinguish the consequences which are reduced to relations over smaller sets of points.

A nontrivial relation Q^τ is called ***proper consequence*** of relation R^δ if τ is a proper subset of δ, i.e., $\tau \subset \delta$, and relation $Q^\tau \times S^{\delta\setminus\tau}$ is consequence of R^δ.

There are relations without proper consequences and these relations are most fundamental for a given number of points k. We call such relations ***prime***.

If relation R^δ has proper consequences $R^{\delta_1}, \ldots, R^{\delta_m}$ we can construct its ***canonical decomposition***

$$R^\delta = PR^\delta \bigcap \left(\bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta\setminus\delta_i} \right) \right), \tag{2}$$

where the factor PR^δ, which we call the ***principal factor***, is defined as

$$PR^\delta = R^\delta \bigcup \left(S^\delta \setminus \bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta\setminus\delta_i} \right) \right).$$

The principal factor is the relation of maximum "freedom", i.e., closest to the trivial relation but sufficient to restore R^δ in combination with the proper consequences.

If the principal factor in canonical decomposition (2) is trivial, then R^δ can be fully reduced to relations over smaller sets of points. We call a relation R^δ ***reducible***, if it can be represented in the form

$$R^\delta = \bigcap_{i=1}^{m} \left(R^{\delta_i} \times S^{\delta\setminus\delta_i} \right),$$

where all R^{δ_i} are proper consequences of R^{δ}. For brevity we will omit the trivial multipliers in intersections and write in the subsequent sections expressions like $\bigcap_{i=1}^{m} R^{\delta_i}$ instead of $\bigcap_{i=1}^{m}\left(R^{\delta_i} \times S^{\delta\setminus\delta_i}\right)$.

We see how to impose the structure of simplicial complex on an amorphous set of points $X = \{x_0, x_1, \ldots\}$ via a relation R^X. The maximal simplices of Δ must correspond to the irreducible components of the relation R^X. Now we can evolve — starting only with a set of points and a relation on it (in fact, we simply identify dimensions of the relation with the points) — the standard tools of the algebraic topology like homology, cohomology, etc.

We wrote a program in ***C*** implementing the above constructions and manipulations with them. Below we illustrate application of the program to analysis of Conway's Game of Life [6] and some of the Wolfram's elementary cellular automata [7].

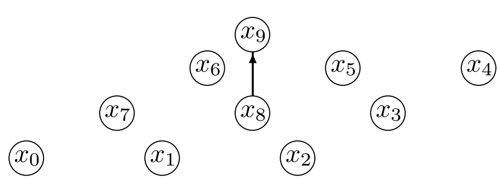
A few words are needed about computer implementation of relations. To specify a k-ary relation R^k we should mark its points within the k-dimensional hypercube S^k, i.e., define a *characteristic function* $\chi : S^k \to \{0,1\}$, with $\chi(\mathbf{s}) = 1$ or 0 according as $\mathbf{s} \in R^k$ or $\mathbf{s} \notin R^k$. Here $\mathbf{s} = (s_0, s_1, \ldots, s_{k-1})$ is a point of the hypercube. The simplest way to implement the characteristic function is to enumerate all the q^k hypercube points in some standard, e.g., lexicographic order:

s_0	s_1	$\ldots$	s_{k-2}	s_{k-1}	i_{ord}
0	0	$\ldots$	0	0	0
1	0	$\ldots$	0	0	1
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$
$q-2$	$q-1$	$\ldots$	$q-1$	$q-1$	q^k-2
$q-1$	$q-1$	$\ldots$	$q-1$	$q-1$	q^k-1

Then the relation can be represented by a string of q^k bits. We call this string *bit table* of relation. Symbolically $\mathrm{BitTable}\,[\,i_{ord}] := \left(\mathbf{s} \in R^k\right)$. Note that $\mathbf{s}$ is a ("little-endian") representation of the number i_{ord} in the base q. Most manipulations with relations are reduced to very efficient bitwise computer commands. Of course, symmetric or sparse (or, vice versa, dense) relations can be represented in a more economical way, but these are technical details of implementation.

3 Conway's Game of Life

The local rule of the cellular automaton ***Life*** is defined over the 10-set $\delta = \{x_0, \ldots, x_9\}$:



Here the point x_9 is the next time step of the point x_8. The state set S is $\{0,1\}$. The local rule can be represented as a relation $R^{\delta}_{\boldsymbol{Life}}$ on the 10-dimensional

hypercube S^δ. By definition, the hypercube element belongs to the relation of the automaton ***Life***, i.e., $(x_0, \ldots, x_9) \in R^\delta_{\boldsymbol{Life}}$, in the following cases:

1. $\left(\sum_{i=0}^{7} x_i = 3\right) \wedge (x_9 = 1)$,
2. $\left(\sum_{i=0}^{7} x_i = 2\right) \wedge (x_8 = x_9)$,
3. $x_9 = 0$, if none of the above conditions holds.

The number of elements of $R^\delta_{\boldsymbol{Life}}$ is $\left|R^\delta_{\boldsymbol{Life}}\right| = 512$. The relation $R^\delta_{\boldsymbol{Life}}$, as is the case for any cellular automaton, is *functional*: the state of x_9 is uniquely determined by the states of other points. The state set $S = \{0, 1\}$ can be *additionally* endowed with the structure of the field $\mathbb{F}_2$. We accompany the below analysis of the structure of $R^\delta_{\boldsymbol{Life}}$ by description in terms of polynomials from $\mathbb{F}_2[x_0, \ldots, x_9]$. This is done only for illustrative purposes and for comparison with the Gröbner basis method. In fact, we transform the relations to polynomials only for output. This is done by computationally very cheap Lagrange interpolation generalized to the multivariate case. In the case $q = 2$, the polynomial which set of zeros corresponds to a relation is constructed uniquely. If $q = p^n > 2$, there is a freedom in the choice of nonzero values of constructed polynomial, and the same relation can be represented by many polynomials.

The polynomial representing $R^\delta_{\boldsymbol{Life}}$ takes the form

$$P_{\boldsymbol{Life}} = x_9 + x_8\left\{\sigma_7 + \sigma_6 + \sigma_3 + \sigma_2\right\} + \sigma_7 + \sigma_3, \tag{3}$$

where $\sigma_k \equiv \sigma_k(x_0, \ldots, x_7)$ is the kth *elementary symmetric polynomial* defined for n variables $x_0, \ldots, x_{n-1}$ by the formula:

$$\sigma_k(x_0, \ldots, x_{n-1}) = \sum_{0 \le i_0 < i_1 < \cdots < i_{k-1} < n} x_{i_0} x_{i_1} \cdots x_{i_{k-1}}.$$

The relation $R^\delta_{\boldsymbol{Life}}$ is reducible. It decomposes into two equivalence classes (with respect to the permutations of the points $x_0, \ldots, x_7$) of relations defined over 9 points:

1. Eight relations $R_1^{\delta\setminus\{x_i\}}, \quad 0 \le i \le 7$.
   Their polynomials $P_1^i(x_0, \ldots, \widehat{x_i}, \ldots, x_7, x_8, x_9)$ take the form

   $$P_1^i = x_8 x_9\left\{\sigma_6^i + \sigma_5^i + \sigma_2^i + \sigma_1^i\right\} + x_9\left\{\sigma_6^i + \sigma_2^i + 1\right\} + x_8\left\{\sigma_7^i + \sigma_6^i + \sigma_3^i + \sigma_2^i\right\},$$

   $$\sigma_k^i \equiv \sigma_k(x_0, \ldots, \widehat{x_i}, \ldots, x_7). \tag{4}$$

2. One relation $R_2^{\delta\setminus\{x_8\}}$ with polynomial $P_2^8(x_0, \ldots, x_7, x_9)$:

   $$P_2^8 = x_9\left\{\sigma_7 + \sigma_6 + \sigma_3 + \sigma_2 + 1\right\} + \sigma_7 + \sigma_3, \qquad \sigma_k \equiv \sigma_k(x_0, \ldots, x_7). \tag{5}$$

The relation $R^{\delta}_{\boldsymbol{Life}}$ has the following decomposition

$$R^{\delta}_{\boldsymbol{Life}} = R_2^{\delta\setminus\{x_8\}} \bigcap \left(\bigcap_{k=0}^{6} R_1^{\delta\setminus\{x_{i_k}\}} \right), \tag{6}$$

where $(i_0, \ldots, i_6)$ are any 7 different indices from the set $(0, \ldots, 7)$.

We see that the rule of ***Life*** is defined on 8-dimensional space-time simplices. Of course, this interpretation is based on the concepts of the abstract combinatorial topology and differs from the native interpretation of the game of ***Life*** as a (2+1)-dimensional lattice structure.

The relations $R_1^{\delta\setminus\{x_i\}}$ and $R_2^{\delta\setminus\{x_8\}}$ are irreducible but not prime, i.e., they have proper consequences.

The relation $R_1^{\delta\setminus\{x_i\}}$ has two classes of 7-dimensional consequences:

1. Seven relations $R_{1.1}^{\delta\setminus\{x_i,x_j\}}$ with polynomials

$$\begin{aligned} P_{1.1}^{ij}\left(x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_7, x_8, x_9\right) = & \\ x_8x_9\left\{\sigma_6^{ij} + \sigma_5^{ij} + \sigma_4^{ij} + \sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1\right\} & \\ +x_9\left\{\sigma_6^{ij} + \sigma_5^{ij} + \sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1\right\}, & \quad (7) \\ \sigma_k^{ij} \equiv \sigma_k\left(x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_7\right). & \end{aligned}$$

2. One relation $R_{1.2}^{\delta\setminus\{x_i,x_8\}}$ with polynomial

$$P_{1.2}^{i}\left(x_0, \ldots, \widehat{x_i}, \ldots, x_7, x_9\right) = x_9\left\{\sigma_7^i + \sigma_6^i + \sigma_5^i + \sigma_3^i + \sigma_2^i + \sigma_1^i + 1\right\}. \tag{8}$$

The 8-dimensional relation $R_2^{\delta\setminus\{x_8\}}$ has one class of 7-dimensional consequences. This class contains 8 already obtained relations $R_{1.2}^{\delta\setminus\{x_i,x_8\}}$ with polynomials (8).

Continuing the process of construction of decompositions and proper consequences we come finally to the prime relations $R^{\delta_{i_0i_1i_2i_3}}$ defined over 4-simplices $\delta_{i_0i_1i_2i_3} = \{x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}, x_9\}$, where $i_k \in \{0, 1, \ldots, 7\}$ and $i_0 < i_1 < i_2 < i_3$. The polynomials of these relations take the form

$$P^{i_0,i_1,i_2,i_3} = x_9\sigma_4\left(x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}\right) \equiv x_9x_{i_0}x_{i_1}x_{i_2}x_{i_3}. \tag{9}$$

Substituting (9) in (4), (5), (7), and (8) (this is a *purely polynomial* simplification) we have finally the following polynomial form of the system of relations valid for the ***Life*** rule:

$$x_8x_9\left\{\sigma_2^i + \sigma_1^i\right\} + x_9\left\{\sigma_2^i + 1\right\} + x_8\left\{\sigma_7^i + \sigma_6^i + \sigma_3^i + \sigma_2^i\right\} = 0, \tag{10}$$

$$x_9\left\{\sigma_3 + \sigma_2 + 1\right\} + \sigma_7 + \sigma_3 = 0, \tag{11}$$

$$\left(x_8x_9 + x_9\right)\left\{\sigma_3^{ij} + \sigma_2^{ij} + \sigma_1^{ij} + 1\right\} = 0, \tag{12}$$

$$x_9\left\{\sigma_3^i + \sigma_2^i + \sigma_1^i + 1\right\} = 0, \tag{13}$$

$$x_9x_{i_0}x_{i_1}x_{i_2}x_{i_3} = 0. \tag{14}$$

Relations (14) have a simple interpretation: if the point x_9 is in the state 1, then at least one of any four points surrounding the center x_8 must be in the state 0.

The above analysis of the relation $R^{\delta}_{\boldsymbol{Life}}$ takes < 1 sec on a 1.8GHz AMD Athlon notebook with 960Mb.

To compute the Gröbner basis we must add to polynomial (3) ten polynomials

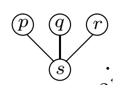
$$x_i^2 + x_i, \ i = 0, \ldots, 9, \tag{15}$$

expressing the relation $x^{p^n} = x$ valid for all elements of any finite field $\mathbb{F}_{p^n}$.

Computation of the Gröbner basis over $\mathbb{F}_2$ with the help of Maple 9 gives the following. Computation for the pure lexicographic order with the variable ordering $x_9 \succ x_8 \succ \cdots \succ x_0$ remains initial polynomial (3) unchanged, i.e., does not give any additional information. The pure lexicographic order with the variable ordering $x_0 \succ x_1 \succ \cdots \succ x_9$ gives relations (10)—(14) (modulo several polynomial reductions violating the symmetry of polynomials). The computation takes 1 h 22 min. Computation for the degree-reverse-lexicographic order also gives relations (10)—(14) (with the above reservation). The times are 51 min for the variable ordering $x_0 \succ x_1 \succ \cdots \succ x_9$, and 33 min for the ordering $x_9 \succ x_8 \succ \ldots \succ x_0$.

4 Elementary Cellular Automata

Simplest binary, nearest-neighbor, one-dimensional cellular automata were called *elementary cellular automata* by S. Wolfram, who has extensively studied their properties [7]. A large collection of results concerning these automata is presented in the Wolfram's online atlas [8]. In the exposition below we use Wolfram's notations and terminology. The elementary cellular automata are simpler than the ***Life***, and we may pay more attention to the topological aspects of our approach.

Local rules of the elementary cellular automata are defined on the 4-set $\delta = \{p, q, r, s\}$ which can be pictured by the icon 

. A local rule is a binary function of the form $s = f(p, q, r)$. There are totally $2^{2^3} = 256$ local rules, each of which can be indexed with an 8-bit binary number.

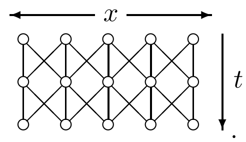
Our computation with relations representing the local rules shows that the total number 256 of them is divided into 118 reducible and 138 irreducible relations. Only two of the irreducible relations appeared to be prime, namely, the rules 105 and 150[4] in Wolfram's numeration.[5]

We consider the elementary automata on a space-time lattice with integer coordinates (x, t), i.e., $x \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ or $x \in \mathbb{Z}_m$ (spatial m-periodicity), $t \in \mathbb{Z}^* = \{0, 1, \ldots\}$. We denote a state of the point on the lattice

[4] They are represented by the linear polynomial equations $p + q + r + s + 1 = 0$ and $p + q + r + s = 0$ for the rules 105 and 150, respectively.

[5] Wolfram prefers "big-endian" representation of binary numbers.

by $u(x,t) \in S = \{0,1\}$. Generally the points are connected as is shown on the 5×3 fragment of the lattice



There are no horizontal ties due to the fundamental property of cellular automata — the states of points at a given temporal layer are independent.

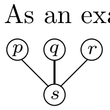
Applying our approach we see that some automata with reducible local relations can be decomposed into automata on disjoint unions of subcomplexes:

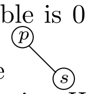
1. Two automata 0 and 255 are defined on disjoint union of vertices.
2. Six automata 15, 51, 85, 170, 204 and 240 are, in fact, disjoint collections of zero-dimensional automata. What we call *zero-dimensional automaton* is spatially zero-dimensional analog of the Wolfram's elementary automaton, i.e., a single cell evolving with time. There are, obviously, four such automata with local relations represented by the bit tables

$$
\begin{aligned}
&1100,\\
&0110,\\
&1001,\\
&0011.
\end{aligned}
\tag{16}
$$

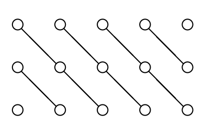
We call the automaton with bit table (16) *oscillating point* since its time evolution consists in periodic changing 0 by 1 and vice versa. It is easy to "integrate" these automata. Their general solutions are respectively

$$
\begin{aligned}
&u(t) = 0,\\
&u(t) = u(0) + t \mod 2, \quad \textit{oscillating point},\\
&u(t) = u(0),\\
&u(t) = 1.
\end{aligned}
\tag{17}
$$

As an example consider the rule 15. The local relation is defined on the set 

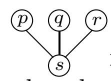
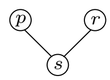
and its bit table is 0101010110101010. This relation is reduced to the relation on the face 

and its bit table 0110 coincides with bit table (16) of the oscillating point. We see that the automaton 15 decomposes into the union of identical zero-dimensional automata on the disconnected lattice



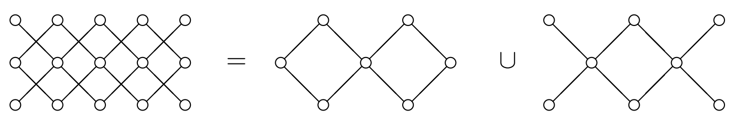
Using (17) we can write the general solution for the automaton 15

$$u(x,t) = u(x-t,0) + t \mod 2.$$

3. Ten automata 5, 10, 80, 90, 95, 160, 165, 175, 245, 250 are decomposed into two identical automata.
As an example let us consider the rule 90. This automaton is distinguished as producing the fractal (of the topological dimension 1 and Hausdorff dimension $\ln 3/\ln 2 \approx 1.58$) known as the Sierpinski sieve, Sierpinski gasket, or Sierpinski triangle. Its local relation on the set  is represented by the bit table 1010010101011010. The relation is reduced to the relation on the face  with the bit table

$$10010110. \tag{18}$$

From the structure of the domain of the reduced relation it is clear that the lattice decomposes into two identical independent lattices as is shown



To find a general solution of the automaton 90 it is convenient to transform bit table (18) to an algebraic relation. It is the linear relation $s + p + r = 0$ and the general solution of the automaton takes the form

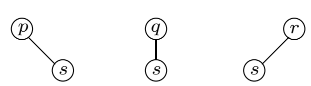
$$u(x,t) = \sum_{k=0}^{t} \binom{t}{k} u(x-t+2k,0) \mod 2.$$

In the above examples we have considered the automata with reducible relations. If a local relation is irreducible but has proper consequences we also, in some cases, can obtain a useful information.

For example, there are 64 automata[6] — both reducible and irreducible — having proper consequencies with the bit table

$$1101 \tag{19}$$

on one or two or three of the following faces



(20)

[6] The full list of these automata in the Wolfram's numeration is 2, 4, 8, 10, 16, 32, 34, 40, 42, 48, 64, 72, 76, 80, 96, 112, 128, 130, 132, 136, 138, 140, 144, 160, 162, 168, 171, 174–176, 186, 187, 190–192, 196, 200, 205, 206, 208, 220, 222–224, 234–239, 241–254.

The algebraic forms of relation (19) on faces (20) are

$$ps + s = 0, \quad qs + s = 0, \quad rs + s = 0,$$

respectively.

Relation (19) is non-functional. Nevertheless, it imposes a severe restriction on the behavior of the automata with such proper consequences. The peculiarities in the behavior are clear visible in the atlas [8], where many results of computations with different initial conditions are pictured. A typical pattern from this atlas is reproduced in Fig. 1, where several evolutions of the automaton 168 are presented. The local relation of the automaton 168 is $pqr + qr + pr + s = 0$. It has the proper consequence $rs + s = 0$. The black and white square cells in Fig. 1 correspond to 1's and 0's, respectively. Note also that the authors of Fig. 1 have used a spatially periodic condition. Their spacial variable is $x \in \mathbb{Z}_{30}$.

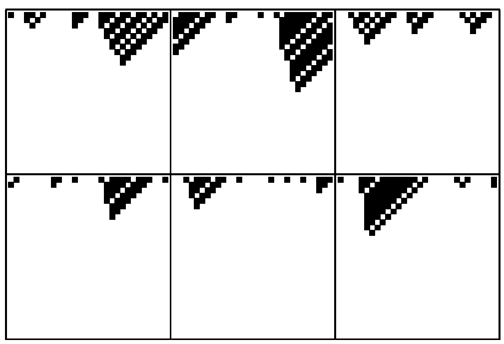


Fig. 1. Rule 168. Several random initial conditions

Relation (19) means that if, say r, as for rule 168, is in the state 1 then s may be in both states 0 or 1, but if the state of r is 0, then the state of s must be 0. Thus the corresponding diagonal or vertical may contain either only 1's, or finite number of initial 1's and then only 0's. The presence of a proper consequence of the form (19) simplifies essentially computation with such automata: after the first appearance of 0, one can set 0's on all points along the corresponding line.

In conclusion, let us present the results of analysis of the automata 30 and 110. These automata are of special interest. The automaton 30 demonstrates chaotic behavior and even used as the random number generator in ***Mathematica.*** The automaton 110 is, like a Turing machine, *universal*, i.e., it is capable of simulating any computational process, in particular, any other cellular automaton. The relations of both automata are irreducible but not prime.

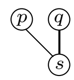
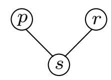
The relation of automaton 30 is

1001010101101010

or in the algebraic form

$$qr + s + r + q + p = 0.$$

It has two proper consequences:

face		
bit table	11011110	11011110
polynomial	$qs + pq + q$	$rs + pr + r$.

The principal factor is

$$1011111101111111 \quad \text{or} \quad qrs + pqr + rs + qs + pr + pq + s + p = 0.$$

The Gröbner basis of automaton 30 in the total degree and reverse lexicographic order is (omitting the trivial polynomials $p^2 + p,\ q^2 + q,\ r^2 + r,\ s^2 + s$)

$$\left\{qr + s + r + q + p,\ qs + pq + q,\ rs + pr + r\right\}.$$

We see that for the rule 30 the Gröbner basis polynomials coincide with ours.

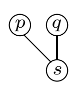
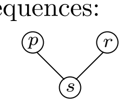
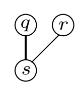
The relation of automaton 110 is

$$1100000100111110 \tag{21}$$

or in the polynomial form

$$pqr + qr + s + r + q = 0.$$

The relation has three proper consequences:

face			
bit table	11011111	11011111	10010111
polynomial	$pqs + qs + pq + q$	$prs + rs + pr + r$	$qrs + s + r + q$.

The principal factor is

$$1111111111111110 \quad \text{or} \quad pqrs = 0.$$

The Gröbner basis of automaton 110 contains different set of polynomials:

$$\left\{prs + rs + pr + r,\ qs + rs + r + q,\ qr + rs + s + q,\ pr + pq + ps\right\}.$$

The system of relations defined by the Gröbner basis is:

$$\begin{aligned}
R_1^{\{p,r,s\}} &= 11011111 = \left\{prs + rs + pr + r = 0\right\},\\
R_2^{\{q,r,s\}} &= 10011111 = \left\{qs + rs + r + q = 0\right\},\\
R_3^{\{q,r,s\}} &= 10110111 = \left\{qr + rs + s + q = 0\right\},\\
R_4^{\{p,q,r,s\}} &= 1110101110111110 = \left\{pr + pq + ps = 0\right\}.
\end{aligned}$$

5 Conclusions

Let us summarize the main novelties of the paper.

– We have introduced a notion of a *system of discrete relations on an abstract simplicial complex*. Such a system can be interpreted as
  - a natural generalization of the notion of cellular automaton;
  - a set-theoretic analog of a system of polynomial equations.
– After introducing appropriate definitions, we have developed and implemented algorithms for
  - *compatibility analysis* of a system of discrete relations;
  - constructing *canonical decompositions* of discrete relations.
– We have proposed a regular way to impose *topology on an arbitrary discrete relation* via its canonical decomposition: identifying *dimensions* of the relation with *points* and *irreducible components* of the relation with *maximal simplices*, we define the structure of an abstract simplicial complex on the relation under consideration.
– Applying the above technique to some cellular automata — a special case of systems of discrete relations — we have obtained some new results. Most interesting of them, in our opinion, is demonstration of how the presence of non-trivial *proper consequences* may determine the global behavior of an automaton.

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